

# Chiral condensates and QCD vacuum in two dimensions

H. R. Christiansen \*

*Centro Brasileiro de Pesquisas Fisicas. CBPF - DCP  
Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil.*

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## Abstract

We analyze the chiral symmetries of flavored quantum chromodynamics in two dimensions and show the existence of chiral condensates within the path-integral approach. The massless and massive cases are discussed as well, for arbitrary finite and infinite number of colors. Our results put forward the question of topological issues when matter is in the fundamental representation of the gauge group.

## 1 Introduction

Correlation functions are key quantities in the understanding of nonperturbative QCD and hadron physics. Basically, because the same functions can be considered in terms of fundamental QCD fields or in terms of physical intermediate states. Using hadronic phenomenological data, one can extract valuable information of the underlying structure of the theory. For example, the difference between vector and axial correlators is entirely due to the chiral asymmetry of the QCD vacuum [1]. Thus, in order to get some insight into the complex structure of the QCD ground state one should consider possible mechanisms for chiral symmetry breaking and condensate formation. In general, these correlators are defined as the vacuum expectation values (v.e.v.) of the product of operators in different space-time points. A nonvanishing v.e.v. of quarks composites can be understood as a result of the condensation of pairs of particles and holes. The strong attractive interaction of quarks and the low energy cost of creating a massless pair are responsible for the appearance of a condensate. Further, the

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\*Electronic address: hugo@cat.cbpf.br

Fourier transform of correlators shows that there is a momentum transfer flowing between the composite operators, making apparent the connection between them by means of the vacuum fields. On the other hand, conservation of the quantum numbers of the vacuum implies a net chiral charge of such correlators. Hence, these quantities are particularly useful in connection to both spontaneous and dynamical chiral symmetry breaking, as well as to shed some light on crucial aspects of quantum chromodynamics such as its topological structure.

Two-dimensional models like quantum chromodynamics in two dimensions (QCD<sub>2</sub>), are convenient frameworks to discuss these kind of phenomena since they present the basic aspects of the fundamental four dimensional theories (such as the existence of nontrivial topological sectors and chirality properties) and, furthermore, analytical results can be generally obtained.

In the present work we analyze the chiral symmetries of the QCD<sub>2</sub> ground state by means of fermionic local correlators. Using a path-integral approach which is very appropriate to handle non-Abelian gauge theories, we calculate vacuum expectation values of products of local bilinears  $\bar{\psi}(x)\psi(x)$  in two-dimensional quantum chromodynamics with flavor. In this framework, we show that certain multipoint chiral condensates are nonvanishing, a result which signals the dynamical breakdown of the full  $U(N_f) \times U(N_f)$  chiral symmetry group down to a smaller subgroup. However, the elementary mass term has a zero v.e.v. in the massless model, a result which is compatible with a vanishing isosinglet chiral anomaly and which is consistent with Coleman's theorem [2]. Thus, the isosinglet axial current is conserved in two-dimensional massless QCD.

In this respect, we address the existence of an elementary chiral condensate in the large  $N_c$  limit, reported in Refs. [3, 4, 5]. To connect our path-integral approach to these alternative procedures, we work over the massive theory and perform a series expansion in the fermionic mass so as to argue about the topological origin of this nontrivial mass like condensate appearing in the 't Hooft's weak phase. The topological structure of the theory is especially considered and we show the central role played by topologically charged sectors in obtaining nonzero correlators.

## 2 Topology

In two space-time dimensions it is generally assumed that the vanishing of the homotopy group  $\pi_1(SU(N))$  implies that QCD<sub>2</sub> exhibits only a trivial topology when fermions are in the fundamental representation of the gauge group. Hence, no vacuum degeneracy is expected to occur [6]. In contrast, when adjoint matter is considered, the relevant symmetry group becomes  $SU(N)/Z_N$  rather than  $SU(N)$ . This implies that  $\pi_1 \neq 0$  which is responsible for the appearance of  $N$  topologically different sectors and instanton effects become apparent.

Nevertheless, by handling fundamental fermions it can be easily verified that

gauge field configurations lying in the Cartan subalgebra of  $SU(N)$  generate non-trivial topological fluxes. Thus, one should include these topologically charged configurations in the path-integral *also* for fermions in the fundamental representation. It should be noticed however, that while  $Z_N$  is the relevant homotopy group for fermions in both the fundamental or the adjoint representation of the gauge group, instantons are not solutions of QCD<sub>2</sub> in any case. Indeed, in two Euclidean dimensions one needs scalar fields and spontaneous symmetry breaking in order to have finite action solutions. Precisely, one has to arrange scalars so as to break the symmetry of the original gauge group down to its center  $Z_N$  since only in this case topologically nontrivial solutions (classified in homotopy classes associated with the elements of  $\pi_1(SU(N)/Z_N) = Z_N$ ) do exist, independently of the fermion representation one will afterwards choose when adding matter.

Concerning this point, let us stress that in two dimensions the role of instantons is basically played by vortices. In the Abelian case, these vortices are identified with the Nielsen-Olesen vortex solutions of a spontaneously broken Abelian Higgs model [7]. This should be contrasted with real QCD where four-dimensional instantons are regular solutions of the gauge field equations of motion when fermions are absent; as we stated above, in the two-dimensional case, either Abelian or non-Abelian, no regular solutions with topological charge exist unless complete symmetry breaking is achieved via Higgs fields. When these scalars are included, the resulting static, axially symmetric gauge field configurations give a realization in a two-dimensional Euclidean theory, of regular gauge fields carrying a topological charge. These classical configurations are then identified with two-dimensional instantons, to be used in non-perturbative analysis of a Maxwell theory coupled to massless fermions [8]. The same route can be undertaken in the non-Abelian case since the analogous to Nielsen-Olesen vortex solutions have been shown to exist, again for spontaneously broken gauge theories [9]. It means that, in the spirit of the path-integral approach to quantum field theory one cannot forget to include these configurations in the integration domain of the theory-without Higgs fields. Once regular gauge field configurations carrying topological charge are identified, the associated fermion zero modes can be found [10, 11] and then, used to study the formation of fermion condensates.

In order to achieve the path-integration it would be worth performing a decoupling transformation of gauge fields from fermions. It is important that the decoupling operation does not change the fiber bundle in which the Dirac operator is defined, hence we start by decomposing every gauge field belonging to the  $n^{th}$  topological sector in the form [12]

$$A_\mu^a(x) = A_\mu^{a(n)} + a_\mu^a \quad (1)$$

where  $A_\mu^{(n)}$  is a classical fixed configuration of  $n^{th}$  flux class and  $a_\mu$  is the path-integral variable which takes into account quantum fluctuations. The quantum field  $a_\mu$  belongs to the trivial topological sector and can then be decoupled from

fermions by a chiral rotation yielding a Fujikawa jacobian [13]. Thus, the integration measure must be only defined on the  $n = 0$  sector.

Topological gauge field configurations and the corresponding zero-modes of the Dirac equation play a central role in calculations involving fermion composites. As in the Abelian case, two-dimensional gauge field configurations  $A_\mu^{(n)}$  carrying a topological charge  $n \in Z_N$  can be found for the  $SU(N)$  case. The relevant homotopy group in this case is  $Z_N$  and not  $Z$  as in the  $U(1)$  case [14]. Taking  $g_n$  in the Cartan subgroup of the gauge group we can write a gauge field configuration belonging to the  $n^{\text{th}}$  topological sector in the form

$$A_\mu^{(n)} = iA(|z|) g_n^{-1} \partial_\mu g_n \quad (2)$$

with  $A(0) = 0$  and  $\lim_{|z| \rightarrow \infty} A(|z|) = -1$ , where  $z = x_0 + ix_1$ .

Zero modes of the Dirac operator in the background of such non-Abelian vortices, have been analyzed in [11]. The outcome is that for topological charge  $n > 0$  ( $n < 0$ ) there are  $Nn$  ( $N|n|$ ) square-integrable zero modes  $\eta_L$  ( $\eta_R$ ) analogous to those arising in the Abelian case. Indeed, one has

$$\eta_R^{(m,i)j} = \begin{pmatrix} z^m h_{ij}(z, \bar{z}) \\ 0 \end{pmatrix}, \quad \eta_L^{(m,i)j} = \begin{pmatrix} 0 \\ \bar{z}^{-m} h_{ij}^{-1}(z, \bar{z}) \end{pmatrix} \quad (3)$$

with

$$h(z, \bar{z}) = \exp(\phi^{(n)}(|z|)M), \quad M = \frac{1}{N} \text{diag}(1, 1, \dots, 1 - N) \quad (4)$$

and  $\phi^{(n)}$  is given by

$$\epsilon_{\mu\nu} \frac{x_\nu}{|z|} \frac{d}{d|z|} \phi^{(n)}(|z|) = A_\mu^{(n)}. \quad (5)$$

Here  $i, j = 1, 2, \dots, N$  and  $m = 0, 1, \dots, |n| - 1$ . The pair  $(m, i)$  labels the  $N|n|$  different zero-modes while  $j$  corresponds to a color projection index.

### 3 Fundamental matter in $QCD_2$

Let us consider two dimensional  $SU(N_c)$  Yang-Mills gauge fields coupled to massless Dirac fermions in the fundamental representation of the group in Euclidean space-time

$$L = \bar{\psi}^q (i\partial_\mu \gamma_\mu \delta^{qq'} + A_{\mu,a} t_a^{qq'} \gamma_\mu) \psi^{q'} + \frac{1}{4g^2} F_{\mu\nu}^a F_{\mu\nu}^a. \quad (6)$$

Here the labels  $a = 1 \dots N_c^2 - 1$ , and  $q = 1 \dots N_c$  are summed over, and the partition function is

$$Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A_\mu \exp[-\int d^2x L]. \quad (7)$$

In order to compute fermionic correlators containing products of local bilinears  $\bar{\psi}\psi(x)$  it will be convenient to decouple fermions from the  $a_\mu$  field through a chiral

rotation within the topologically trivial sector. The choice of an appropriate background like

$$A_+^{a(n)} = 0 \quad (8)$$

is important in order to control the zero-mode problem [17]. Let us start by introducing group-valued fields to represent  $A^{(n)}$  and  $a_\mu$

$$a_+ = iu^{-1}\partial_+u \quad (9)$$

$$a_- = id(v\partial_-v^{-1})d^{-1} \quad (10)$$

$$A_-^{(n)} = id\partial_-d^{-1}. \quad (11)$$

In terms of these fields the fermion determinat can be suitably factorized in an arbitrary gauge by repeated use of the Polyakov-Wiegmann identity [18], resulting in

$$\det \not{D}[A^{(n)} + a] = \mathcal{N} \det \not{D}[A^{(n)}] e^{-S_{eff}[u,v;A^{(n)}]} \quad (12)$$

where

$$\begin{aligned} S_{eff}[u, v; A^{(n)}] = & W[u, A^{(n)}] + W[v] + \frac{1}{4\pi} tr_c \int d^2x (u^{-1}\partial_+u) d(v\partial_-v^{-1})d^{-1} \\ & + \frac{1}{4\pi} tr_c \int d^2x (d^{-1}\partial_+d)(v\partial_-v^{-1}). \end{aligned} \quad (13)$$

Here  $W[u, A^{(n)}]$  is the gauged Wess-Zumino-Witten action which in this case takes the form

$$W[u, A^{(n)}] = W[u] + \frac{1}{4\pi} tr_c \int d^2x (u^{-1}\partial_+u)(d\partial_-d^{-1}) \quad (14)$$

and  $W[u]$  is the usual WZW action.

Once the determinant has been written in the form (12), one can work with any gauge choice. The partition function shows the following structure

$$\begin{aligned} Z = & \sum_n \det(\not{D}[A^{(n)}]) \int \mathcal{D}a_\mu \Delta_{FP} \delta(F[a]) \\ & \exp \left( -S_{eff}[A^{(n)}, a_\mu] - \frac{1}{4g^2} \int d^2x F_{\mu\nu}^2[A^{(n)}, a_\mu] \right) \end{aligned} \quad (15)$$

where  $\Delta_{FP} \delta(F[a])$  comes from the gauge fixing.

## Condensates of fundamental matter fields

As it happens in the Abelian case, the partition function of two dimensional quantum chromodynamics only picks a contribution from the trivial sector because  $\det(\not{D}[A^{(n)}]) = 0$  for  $n \neq 0$  (see eq.(15)). In contrast, various correlation functions become nontrivial precisely for  $n \neq 0$  thanks to the zero-mode contributions when Grassman integration is performed.

In order to obtain a general expression for multipoint local correlators let us work with the gauge choice given in eq.(8). Now, the Dirac equation takes the form

$$\mathcal{D}[A^{(n)} + a] \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0 & u^{-1}i\partial_+ \\ dvd^{-1}D_-[A^{(n)}] & 0 \end{pmatrix} \begin{pmatrix} \zeta_+ \\ \zeta_- \end{pmatrix} \quad (16)$$

where  $\zeta$  is defined by

$$\psi_+ = dvd^{-1}\zeta_+, \quad \psi_- = u^{-1}\zeta_- \quad (17)$$

Thus, the interaction Lagrangian in the  $n^{th}$  flux sector reads

$$L = \bar{\psi} \mathcal{D}[A^{(n)} + a]\psi = \zeta_+^* \mathcal{D}_-[A^{(n)}]\zeta_+ + \zeta_-^* i\partial_+\zeta_- \quad (18)$$

which we will write as  $\bar{\zeta} \widetilde{D}[A^{(n)}]\zeta$ . In terms of these new fields, the elementary bilinear  $\bar{\psi}\psi$  takes the form

$$\bar{\psi}\psi = \zeta_-^* u d v d^{-1} \zeta_+ + \zeta_+^* d v^{-1} d^{-1} u^{-1} \zeta_- \quad (19)$$

Notice that the fermionic jacobian associated with eq.(17) is just the effective action defined in the previous section by eq.(13). Hence, arbitrary non-Abelian correlators of fundamental fermions can be put down as

$$\begin{aligned} \langle \bar{\psi}\psi(x^1) \dots \bar{\psi}\psi(x^l) \rangle &= \sum_n \int \mathcal{D}u \mathcal{D}v \Delta_{FP} \delta(F[a_\mu]) \exp[-S_{eff}(A^{(n)}, u, v)] \\ &\int \mathcal{D}\bar{\zeta} \mathcal{D}\zeta \exp(\bar{\zeta} \begin{pmatrix} 0 & i\partial_+ \\ D_-[A^{(n)}] & 0 \end{pmatrix} \zeta) \\ &\left( B^{q_1 p_1}(x^1) \dots B^{q_l p_l}(x^l) \zeta_-^{*q_1} \zeta_+^{p_1}(x^1) \dots \zeta_-^{*q_l} \zeta_+^{p_l}(x^l) + B^{q_1 p_1}(x^1) \dots \right. \\ &B^{-1q_l p_l}(x^l) \zeta_-^{*q_1} \zeta_+^{p_1}(x^1) \dots \zeta_+^{*q_l} \zeta_-^{p_l}(x^l) + B^{q_1 p_1}(x^1) \dots \\ &B^{-1q_{l-1} p_{l-1}}(x^{l-1}) B^{-1q_l p_l}(x^l) \zeta_-^{*q_1} \zeta_+^{p_1}(x^1) \dots \zeta_+^{*q_{l-1}} \zeta_-^{p_{l-1}}(x^{l-1}) \zeta_+^{*q_l} \zeta_-^{p_l}(x^l) \\ &\left. + \dots + B^{-1q_1 p_1}(x^1) \dots B^{-1q_l p_l}(x^l) \zeta_+^{*q_1} \zeta_-^{p_1}(x^1) \dots \zeta_+^{*q_l} \zeta_-^{p_l}(x^l) \right) \end{aligned} \quad (20)$$

where the group-valued field  $B$  is given by  $B = u d v d^{-1}$ . Note that this is a general and completely decoupled expression for fermionic correlators, which shows that the simple product found in the Abelian case becomes here an involved sum due to color couplings.

The introduction of a flavor index implies additional degrees of freedom which result in  $N_f$  independent fermionic field variables. Consequently, the growing number of Grassman (numeric) differentials calls for additional Fourier coefficients in the integrand. Dealing with  $N_f$  fermions coupled to the gauge field, leads to the fermionic jacobian computed for one flavor to the power  $N_f$ , the bosonic measure remaining the same. As we have previously explained, the Dirac operator has  $|n|N_c$  zero modes in the  $n^{th}$  topological sector, implying that more

fermion bilinears will be needed in order to obtain a nonzero fermionic path-integral. Moreover, since flavor comes together with a factor  $N_f$  on the number of Grassman coefficients, the minimal nonzero product of fermion bilinears in the  $n^{th}$  sector requires of  $|n|N_cN_f$  insertions.

Since the properties of all these topological configurations are given by those in the torus of  $SU(N_c)$ , one can easily extend the results obtained in the Abelian case. In particular, the chirality of the zero modes is dictated by the same index theorem found in the Abelian theory, this implying that in sector  $n > 0$  ( $n < 0$ ) every zero mode has positive (negative) chirality. In this way, the right (left) chiral projections of the minimal nonzero fermionic correlators can be easily computed.

Since equations became a bit involved, in order to illustrate simple flavored extensions of expression (20), we just consider  $N_f = 2$  for two colors. The minimal fermionic correlator then looks

$$\begin{aligned} & \sum_n \langle \bar{\psi}_+^{1,1} \psi_+^{1,1}(x^1) \bar{\psi}_+^{1,2} \psi_+^{1,2}(x^2) \bar{\psi}_+^{2,1} \psi_+^{2,1}(y^1) \bar{\psi}_+^{2,2} \psi_+^{2,2}(y^2) \rangle_n = \\ & \frac{1}{Z^{(0)}} \sum_{p,q,r,s}^{N_c=2} \prod_{k=1}^2 \int_{gf} \mathcal{D}u \mathcal{D}v J_B e^{-S_{Eff}^{(1)}(u,v,d)} B_k^{1,pq}(x^k) B_k^{2,rs}(y^k) \times \\ & \int \mathcal{D}\bar{\zeta}_k \mathcal{D}\zeta_k e^{\int \bar{\zeta}_k \widetilde{D}[A^{(1)}] \zeta_k} \bar{\zeta}_+^{p,k} \zeta_+^{q,k}(x^k) \bar{\zeta}_+^{r,k} \zeta_+^{s,k}(y^k). \end{aligned} \quad (21)$$

where

$$B_k^{q,p_i l_i}(x) = u^{p_i q}(x) (dv d^{-1})^{q l_i}(x), \quad (22)$$

$\widetilde{D}[A^{(n)}]$  is the Dirac operator as defined in eq.(18), and  $\bar{\zeta}_+$  stands for  $\zeta_-^*$ . We have employed the notation  $Z^{(0)}$  for the partition function to emphasize that it is completely determined within the  $n = 0$  sector, see eq.(15). The  $gf$  subindex stands for the gauge fixing. The action  $S_{Eff}^{(n)}(u, v, d) = N_f S_{WZW}(u, v, d) + S_{Maxwell}(u, v, d)$  is given by the full gluon field  $A^{(n)}(d) + a(u, v)$ , and yields a high order Skyrme-type lagrangian [15].

The fermionic path-integral can be easily performed, amounting to a product of the eigenfunctions discussed in the sections above, as follows

$$\begin{aligned} & \int \mathcal{D}\bar{\zeta}_k \mathcal{D}\zeta_k e^{\int \bar{\zeta}_k \widetilde{D}[A^{(1)}] \zeta_k} \bar{\zeta}_+^{p,k} \zeta_+^{q,k}(x^k) \bar{\zeta}_+^{r,k} \zeta_+^{s,k}(y^k) = \det \nu(\widetilde{D}[A^{(1)}]) \times \\ & \left( -\bar{\eta}_+^{(0,1)p,k} \eta_+^{(0,1)q,k}(x^k) \bar{\eta}_+^{(0,2)r,k} \eta_+^{(0,2)s,k}(y^k) + \bar{\eta}_+^{(0,1)p,k} \eta_+^{(0,2)q,k}(x^k) \right. \\ & \left. \bar{\eta}_+^{(0,2)r,k} \eta_+^{(0,1)s,k}(y^k) - \bar{\eta}_+^{(0,2)p,k} \eta_+^{(0,1)q,k}(x^k) \bar{\eta}_+^{(0,1)r,k} \eta_+^{(0,2)s,k}(y^k) \right. \\ & \left. + \bar{\eta}_+^{(0,2)p,k} \eta_+^{(0,2)q,k}(x^k) \bar{\eta}_+^{(0,1)r,k} \eta_+^{(0,1)s,k}(y^k) \right). \end{aligned} \quad (23)$$

Here  $\det \nu(\widetilde{D}[A^{(1)}])$  is the determinat of the Dirac operator defined in eq.(18) omitting zero modes, and (e.g.)  $\eta_+^{(0,1)q,k}(x^k)$  is a non-Abelian zero-mode as defined in section 2, with an additional flavor index  $k$ . Concerning the bosonic sector, the presence of the  $F_{\mu\nu}^2$  (Maxwell) term crucially changes the effective dynamics with

respect to that of a pure Wess-Zumino model. One then has to perform approximate calculations to compute the bosonic factor, for example, by linearizing the  $U$  transformation, see [15]; nevertheless, the point relevant to our discussion of obtaining nonzero fermionic correlators is manifest in eq.(23).

## 4 Nonzero $\langle \bar{\psi}\psi \rangle$

As a byproduct, our approach gives  $\langle \bar{\psi}\psi \rangle = 0$  in every flux sector including  $n = 0$ . This is consistent with Coleman's theorem which prohibits the spontaneous breakdown of any continuous symmetry in two dimensions. Furthermore, since, in contrast to the Abelian case, QCD<sub>2</sub> presents no (isosinglet) axial anomaly which could give rise to a nonzero mass like condensate  $\langle \bar{\psi}\psi \rangle$ .

By the other side the existence of nonvanishing elementary condensates have been discussed within alternative scenarios for one flavor QCD<sub>2</sub> [3, 4, 6]. One is based on the assumption of an infinite number of colors while the other focuses matter in the adjoint representation (see also next section). This reminds one the outcome for massless two-dimensional QED [19, 20].

It is well known that in the Abelian case a multipoint composite receives contributions from different topological sectors. In particular, one can obtain the value of the elementary scalar condensate from a two point correlator since the term  $\langle \bar{\psi}_+\psi_+(x)\bar{\psi}_-\psi_-(y) \rangle$  factorizes as  $\langle \bar{\psi}_+\psi_+(x) \rangle \cdot \langle \bar{\psi}_-\psi_-(y) \rangle$ , the two factors being equal each other and nonzero [19]. Then, by means of the chiral decomposition  $\langle \bar{\psi}\psi(x) \rangle = \langle \bar{\psi}_+\psi_+(x) \rangle + \langle \bar{\psi}_-\psi_-(x) \rangle$  one constructs the condensate from the trivial topological sector although  $\langle \bar{\psi}_\pm\psi_\pm(x) \rangle$  come exclusively from flux classes  $\pm 1$  respectively [20]. The nonzero result indicates the breakdown of the chiral symmetry and it is known to be due to the  $U(1)$  anomaly.

In the non-Abelian theory in turn, the result  $\langle \bar{\psi}\psi \rangle = 0$  thereby entails no anomalous chiral symmetry in the singlet channel. For a finite number of colors this is in agreement with independent analytical calculations based on operator product expansion and dispersion relations [3] and canonical quantization on the light-cone front [4]. Though, in contrast to ours, these previous analysis have been only performed in the trivial topological sector and, further, it has been assumed that cluster decomposition holds.

In the large  $N_c$  limit, cluster decomposition takes place so that all v.e.v. can be reduced to a product of elementary scalar densities, i.e.  $\langle AB \rangle = \langle A \rangle \langle B \rangle + O(1/N_c)$ . On the other hand, for very separated composites factorization also holds. Outside any of these asymptotic situations we have shown which are the actual results for arbitrary values of both color and relative positions. Notice that only for an infinite number of colors the Berezinskii-Kosterlitz-Thouless (BKT) behaviour [21] of such an elementary fermion correlator is compatible with a nonzero outcome for  $\langle \bar{\psi}\psi \rangle$ . The BKT effect can be shown by means of QCD sum rules giving a nonzero condensate when the weak coupling regime is considered.



In fact, bosonization rules show that for large distances [3]

$$\langle \bar{\psi}_+ \psi_+(x) \bar{\psi}_- \psi_-(y) \rangle \sim |x - y|^{-1/N_c} \quad (24)$$

implying that for large but finite  $N_c$  it smears away softly. Actually, this effect cannot be seen unless the  $M \rightarrow 0$  limit is taken at the end of the calculation in the *massive* theory. Moreover, the BKT behavior is a very especial result that only takes place under the  $g^2 N_c = \text{const}$  condition, in a particular asymptotic direction. The limit  $N_c \rightarrow \infty$ ,  $g \rightarrow 0$  and  $M \rightarrow 0$  is understood -but only- provided  $M \gg g \sim 1/\sqrt{N_c} \rightarrow 0$  ('t Hooft regime). Obviously this establishes a very particular phase transition wherever one of the constraints does not hold together with the others. Furthermore, the value of the correlator also depends on the choice of the approximation to the limit  $N_c \rightarrow \infty$ ,  $|x - y| \rightarrow \infty$ , see eq.(24).

## 5 The massive case

As we have mentioned, this BKT phenomenon only takes place provided the chiral symmetry is explicitly broken from the beginning by means of a quark mass term. One could then say that after the chiral weak phase is reached in the massive theory, the symmetry then keeps in a broken phase, but instead, it happens dynamically. The physics changes, the vacuum being dressed non-perturbatively by means of purely planar diagrams [22]. The spectrum is completely different depending on the relations shown above; in the weak phase there is an infinite number of massive mesons while in the strong coupling phase there are just massless baryons. In the massless non-Abelian theory the 'pion' is also massless but there is no spontaneous symmetry breaking as dictated by Coleman's theorem. Hence, in order to connect this nonzero outcome for the elementary condensate, we should extend our procedure to massive QCD<sub>2</sub>.

Now, the partition function can be put as

$$Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A_\mu \exp\left(-\int d^2x L_{M=0}\right) e^{-M \int d^2x \bar{\psi}\psi}. \quad (25)$$

Actually, the analytical solution of this model is still lacking and it might be a fruitless effort to attack this problem. Nevertheless, for our purposes only small masses need to be examined. Written in this form, it is suggested that we may perform a perturbative expansion in terms of the quark mass [23]. In order to compare the outcome with the results above let us consider one flavor fermions. Then, we can proceed with the minimal 'massive' condensate as follows

$$\langle \bar{\psi}\psi(\omega) \rangle_M = \sum_n \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}a_\mu \exp\left(-\int d^2x L_{(M=0)}^{(n)}\right) \bar{\psi}\psi(\omega) e^{-M \int d^2x \bar{\psi}\psi}. \quad (26)$$

In this fashion, it is apparent that for  $M \neq 0$  this elementary condensate receives contributions from every correlator coming from the massless theory

$$\begin{aligned} \langle \bar{\psi}\psi(\omega) \rangle_M &= \sum_n \langle \bar{\psi}\psi(\omega) \rangle_{M=0}^{(n)} + M \sum_n \int d^2x \langle \bar{\psi}\psi(\omega) \bar{\psi}\psi(x) \rangle_{M=0}^{(n)} + \\ &\frac{1}{2} M^2 \sum_n \int d^2x d^2y \langle \bar{\psi}\psi(\omega) \bar{\psi}\psi(x) \bar{\psi}\psi(y) \rangle_{M=0}^{(n)} + \dots \end{aligned} \quad (27)$$

In terms of the chiral decomposition mentioned above, we write

$$\langle \bar{\psi}\psi(x) \rangle = \langle \bar{\psi}_+\psi_+(x) \rangle + \langle \bar{\psi}_-\psi_-(x) \rangle = \langle s_+(x) \rangle + \langle s_-(x) \rangle \quad (28)$$

As we have seen, the existence of  $nN$  zero modes in topological sector  $n$  implies the vanishing of the first summatory in eq.(27)  $\forall n$ . However, for higher powers of  $M$  it is clear that certain nonzero contributions come into play. As we have explained, the existence of zero modes with a definite chirality (positive for  $n > 0$ , negative for  $n < 0$ ) set the (Grassman) integration rules of v.e.v.'s. Since the fermionic sector has been completely decoupled, the counting of the nonzero pieces simply follows from that of the Abelian case, because the topological structure here can be red out from the torus of the gauge group. To be more specific: the first  $N_c$  powers of  $M$  ( $j = 1 \dots N_c$ ) receive an input from (just) the trivial topological sector. Then, for  $j \geq N_c$ , the  $n = 1$  sector starts contributing together with  $n = 0$ . For powers  $j \geq 2N_c$  the contribution of topological sector  $n = 2$  starts, and the counting follows so on in this way.

Now, it can be easily seen that the number of contributions grows together with the number of colors. Usually, one investigates the full quark condensate  $\langle \bar{q}q(x) \rangle_M$  with the color index summed over; then, one has a sum of  $N_c$  identical nonzero components  $\langle \bar{q}^c q^c(x) \rangle_M$  so that the complete condensate reads  $N_c \langle \bar{q}^c q^c(x) \rangle_M$  (here  $c$  represents any color). As we let  $N_c$  go to infinity the elementary massive condensate does so. On the other hand, since within each nontrivial topological sector the number of zero modes grows also to infinity, one has divergent v.e.v.'s everywhere in the series expansion of eq.(27). Accordingly, high order terms could also produce a nonzero outcome in the chiral limit.

Therefore, it is clear that the limit  $M \rightarrow 0$  becomes matter of a careful analysis; namely, combined with a large number of colors, eq.(27) leaves place enough for a nontrivial elementary condensate even in the chiral limit.

## 6 Summary and discussion

V.e.v.s of an arbitrary number of fermionic bilinears in multiflavor non-Abelian gauge theories in Euclidean two space-time dimensions have been presented in order to discuss chiral symmetry issues of massless  $QCD_2$ .

By means of a path-integral procedure we have shown how topological effects give rise to non trivial correlators for a theory with a vanishing homotopy group,

$\pi_1(SU(N_c)) = 0$ . These results make apparent that the topological structure found in QCD<sub>2</sub> for matter in the fundamental representation is indeed very important. As we have explained, gauge fields lying in the Cartan subalgebra of  $SU(N_c)$  have to be taken into account to find a significant outcome for fermion condensates although in the massless theory the elementary condensate is identically zero.

Let us mention that, in contrast, the elementary mass like condensate is not zero when massless fermions are in the adjoint representation of the gauge group; however it is not an order parameter for the chiral phase transition and it does not breakdown any continuous symmetry. The existence of this condensate is apparent from bosonization rules and it is also expected from its topological properties as has been argued in section 2. With a standard normalization this v.e.v. of Majorana fermions amounts to  $\langle \bar{\Psi}^a \Psi^a \rangle \sim \sqrt{g^2 N_c} N_c$  for a vanishing fermion mass and  $N_c$  not necessarily divergent [6]. This value coincides with the one obtained in the 't Hooft regime for fundamental matter but in this case only for an infinite number of colors [3, 4].

Notice that as  $N_c \rightarrow \infty$  the relevant homotopy group of  $SU(N_c)$  approaches  $Z$ . One could be tempted to conclude that this fact relates the non-Abelian theory in the large  $N_c$  limit to its Abelian counterpart, in the sense that the anomalous chiral symmetry breakdown in the singlet channel; however, the inclusion of a mass term seems to be crucial for just the non-Abelian case.

Our analysis of the massive theory, put forward a possible way out to the BKT phenomenon by means of a path-integral approach and topological considerations. In this way, both the trivial result in the massless theory and the nonzero outcome in the massive one, have gained contact with those emerging from alternative procedures [3, 4, 6]. The discussion emphasizes the important role of the path taken to the chiral phase limit in order to obtain finite results.

In contrast to the alternative approaches mentioned above, our treatment makes apparent the crucial role of topology into the properties of the QCD vacuum, which for this reason, is a suitable scheme to analyze this issue. In spite of the previous discussion, for any finite value of  $N_c$  the axial anomaly in the multiplet channel dynamically breaks the flavor symmetry of QCD<sub>2</sub> for fundamental matter, allowing the existence of nonvanishing correlators for a larger number of points, see e.g. eq.(21). In particular, it implies the circumvention of a decomposition ansatz and the large  $N_c$  approximation becomes unnecessary in order to find nontrivial multipoint correlators. The elementary (massless) condensate appears to be nontrivial just for an infinite number of colors in the chiral limit of the massive theory, a result which comes forth in our treatment too. As we have shown, condensates are completely determined by the topological structure of the theory; however, the question of exactly which topologically nontrivial configuration or master field could saturate the elementary condensate for  $N_c = \infty$  remains unanswered in an analytical way.

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